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Bethe-ansatz type equations for the Fateev–Zamolodchikov spin model

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Abstract. The eigenvalues of the Fateev-Zamolodchikov Z_N invariant model transfer matrix are found for N odd. Their zeros in the complex plane of the rapidity variable are shown to satisfy a set of Bethe-ansatz type equations similar to those obtained for the integrable XXZ chains. The eigenvalue for a filled sea of $(N-1)$-strings gives the free energy found by the matrix inversion method.

1. Introduction

The Fateev-Zamolodchikov model (FZM) [1] is a two-dimensional lattice spin model with N-valued spins on the sites of the lattice, nearest neighbours interactions and global invariance under the Z_N group of discrete rotations in spin space. In [1] it was shown that it is possible to choose the interactions between spins so that the model is self-dual [2] and the Boltzmann weights satisfy the star-triangle equations [3]. In the same paper, the free energy was found by means of the matrix inversion method [4].

The same authors proposed in [5] a conformal field theory with Z_N symmetry group and invariance of the correlation functions under a duality transformation, and conjectured that such theory should describe the scaling limit of the FZM. The conjecture has already been confirmed by several authors [6, 7].

The FZM can be regarded as the non-chiral limit of the self-dual chiral Potts model [8]. Several recursion relations for the transfer matrix of the chiral Potts model were recently found [9, 10], and used to determine the largest eigenvalue of the transfer matrix [11, 12] and the next-to-largest eigenvalue [13]. On the other hand, the non-chiral limit can be handled more easily, because the Boltzmann weights are neatly parametrized in terms of trigonometric functions of a single rapidity variable. This makes it possible, and interesting, to study all the eigenvalues of the FZM transfer matrix and associated quantum spin chain with standard methods of exact integrability. In particular, finite size corrections [14] should reproduce the full spectrum of conformal dimensions predicted in [5].

In this paper we begin the investigation, reducing the problem of finding the eigenvalues of the transfer matrix and the quantum spin chain to a set of Bethe-ansatz type equations. For technical reasons, discussed in section 4, only the case N odd will be considered. In section 2 we discuss some general properties of the FZM. The general form of the eigenvalues is found in section 3 and the Bethe-ansatz type equations

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derived in section 4. In section 5 we conjecture that the ground state of the spin chain should be given by a filled band of $(N-1)$-strings, showing that, with this assumption, one recovers the free energy of the 2D model, originally found in [1].

2. General properties

The spin variables of the model live on the sites of a square lattice and take on the values $n=0,1,\ldots,N-1$. In [1] it was proved that the one-parameter family of Boltzmann weights (BW)

$$\frac{W(n|u)}{W(0|u)}=\prod_{j=1}^{n}\frac{\sin(\pi j/N-\pi/2N-u)}{\sin(\pi j/N-\pi/2N+u)} \tag{1}$$

$$\frac{\bar{W}(n|u)}{\bar{W}(0|u)}=\prod_{j=1}^{n}\frac{\sin(\pi j/N-\pi/N+u)}{\sin(\pi j/N-u)} \tag{2}$$

is a solution of the star-triangle equations. Here u is $\alpha/2N$ of [1] and we will adopt the normalization $W(0|u)=\bar{W}(0|u)=1$. The BW are real non-negative when the rapidity u is the 'physical region' $[0,\pi/2N]$. Notice also the property $W(N+n)=W(n)$, $\bar{W}(N+n)=\bar{W}(n)$. For $N=3$, (1) and (2) simply reduce to the self-dual 3-state Potts model (but for $N>3$ they do *not* give the N-state Potts model).

The FZM can also be seen as the non-chiral limit of the self-dual chiral Potts model [8], whose BW are

$$\frac{W_{pq}(n)}{W_{pq}(0)}=\prod_{j=1}^{n}\frac{b_q-\omega^j a_p}{b_p-\omega^j a_q} \tag{3}$$

$$\frac{\bar{W}_{pq}(n)}{\bar{W}_{pq}(0)}=\prod_{j=1}^{n}\frac{\omega a_p-\omega^j a_q}{b_q-\omega^j b_p} \tag{4}$$

where $\omega=\exp(2\pi \mathrm{i}/N)$ and the (a,b) variables satisfy the constraint

$$a_x^N+b_x^N=\kappa \tag{5}$$

κ a constant, $x=p$ or q. When $\kappa=0$, we parametrize (5) taking

$$a_x=\mathrm{e}^{2\mathrm{i}x} \qquad b_x=\omega^{1/2}\,\mathrm{e}^{2\mathrm{i}x} \tag{6}$$

and set $u=q-p$. Equations (3), (4) reduce then to (1), (2). This observation will be used in section 4.

Since (1), (2) are a solution of the star-triangle equations [3] they can be used to construct a family of commuting transfer matrices

$$T_{n,n'}(u)=\prod_{k=1}^{M}\bar{W}(n_k-n'_k|u)\,W(n_k-n'_{k+1}|u) \tag{7}$$

$$[T(u),T(u')]=0 \qquad \forall u,u'\in\mathbb{C} \tag{8}$$

where M is the number of sites in one row and periodic boundary conditions are understood. Expression (7) is the matrix representation of an operator acting on the complex linear space spanned by the spin configurations $n=|n_1,n_2,\ldots,n_M\rangle$, $n_k=0,1,\ldots,N-1$. It reduces to the identity operator when $u=0$ and to the shift operator when $u=\pi/2N$

$$T(\pi/2N)=S^{-1}=\mathrm{e}^{\mathrm{i}P} \qquad S|n_1,\ldots,n_M\rangle=|n_M,n_1,\ldots,n_{M-1}\rangle.$$

It is convenient to introduce the following set of operators

$$X_k|n_1 \ldots n_k \ldots n_M\rangle = |n_1 \ldots n_k+1 \ldots n_M\rangle \qquad \bmod N$$
$$Z_k|n_1 \ldots n_k \ldots n_M\rangle = \omega^{n_k}|n_1 \ldots n_k \ldots n_M\rangle.$$

Because of global Z_N invariance of the lattice model, $T(u)$ commutes with

$$X = \prod_{k=1}^{M} X_k$$

and since $X^N = 1_{\text{id}}$ we set $X = \exp(2i\pi Q/N)$ where the Z_N charge Q can take the values $Q = 0, 1, \ldots, N-1$. Furthermore $T(u)$ commutes with the 'charge conjugation' operator

$$C|n_1 \ldots n_M\rangle = |N-n_1 \ldots N-n_M\rangle$$

because $W(N-n) = W(n)$ and $\bar{W}(N-n) = \bar{W}(n)$. Since C maps the sector Q into the sector $N-Q$, we conclude that the eigenvalues of $T(u)$ are labelled by Q, and the spectra in the sectors Q and $N-Q$ are identical. Finally we define the associated quantum spin chain Hamiltonian H [15] from the expansion

$$\begin{aligned} T(u) &= 1 - Mu \sum_{n=1}^{N-1} \frac{1}{\sin(n\pi/N)} - uH + \mathrm{O}(u^2) \\ H &= -\sum_{k=1}^{M} \sum_{n=1}^{N-1} \frac{1}{\sin(n\pi/N)} (X_k^n + Z_k^n Z_{k+1}^{-n}). \end{aligned} \tag{9}$$

Equation (9) implies that H has an infinite set of conserved charges in involution.

For small positive u the ground state of H obviously corresponds to the largest eigenvalue of $T(u)$, but when the BW are strictly positive Perron–Frobenius theorem [16] guarantees that there can be no level crossing for the largest eigenvalue of $T(u)$, hence the correspondence extends throughout the physical region. We finally remark that the spin chain Hamiltonian of the self-dual chiral Potts model

$$\begin{aligned} H(\phi) &= -\sum_{k=1}^{M} \sum_{n=1}^{N-1} \alpha_n (X_k^n + Z_k^n Z_{k+1}^{-n}) \\ \alpha_n &= \exp[i(2n-N)\phi/N]/\sin(n\pi/N) \end{aligned} \tag{10}$$

can be reduced to (9) by setting the chirality parameter $\phi = 0$. On the other hand, under the action of the unitary operator

$$U = \prod_{k=1}^{M} Z_k^{-1} \prod_{k=1}^{M} X_k^{-k}$$

$H(\phi)$ transforms as

$$UH(\phi)U^{-1} = -H(\phi - \pi)$$

provided that $M = 0 \bmod N$, so $H(\phi = \pi)$ is unitarily equivalent to $-H$. While the physical properties of H and $-H$ can be very different, the diagonalization of $T(u)$ would give the complete spectrum of (10) at these two distinct points of the chirality parameter.

3. The spectrum of $T(u)$

From now on we restrict our considerations to N odd (the reasons for this choice will be explained in the next section). The distinct BW that appear in (1), (2) are $W(0), W(1), \ldots, W((N-1)/2)$, $\bar{W}(0), \bar{W}(1), \ldots, \bar{W}((N-1)/2)$. We remove the denominators defining a normalized transfer matrix

$$T^N(u)=[g(u)\bar{g}(u)]^M T(u)$$

where

$$g(u)=\prod_{j=1}^{(N-1)/2}\sin\left(\frac{\pi j}{N}-\frac{\pi}{2N}+u\right) \qquad \bar{g}(u)=\prod_{j=1}^{(N-1)/2}\sin\left(\frac{\pi j}{N}-u\right).$$

Each entry of $T^N(u)$ is a product of $(N-1)M$ sines and it has the general form

$$\prod_{k=1}^{(N-1)M}(c_k^{(1)}\,\mathrm{e}^{\mathrm{i}u}+c_k^{(2)}\,\mathrm{e}^{-\mathrm{i}u}).$$

Call $\Lambda(u)$ the eigenvalues of $T(u)$. Owing to (8) the eigenvectors of $T(u)$ do not depend on the rapidity u

$$T(u)|v\rangle=\Lambda(u)|v\rangle \tag{11}$$

so each eigenvalue is a linear combination of matrix elements with coefficients which do not depend on u [3]. Consequently, we must have

$$\Lambda(u)=\left[\frac{1}{g(u)\bar{g}(u)}\right]^M P(\mathrm{e}^{\mathrm{i}u}) \tag{12}$$

$P(\mathrm{e}^{\mathrm{i}u})$ being a Laurent polynomial in $\mathrm{e}^{\mathrm{i}u}$. Furthermore $T(u+\pi)=T(u)$ and the prefactor in (12) is invariant under $u\to u+\pi$ so only even powers appear in P

$$\begin{gathered}P(\mathrm{e}^{\mathrm{i}u})=P_A^B(\mathrm{e}^{2\mathrm{i}u})=c_B\,\mathrm{e}^{2\mathrm{i}uB}+c_{B-1}\,\mathrm{e}^{2\mathrm{i}u(B-1)}\ldots c_{-A}\,\mathrm{e}^{2\mathrm{i}u(-A)}\\ c_B,\,c_{-A}\neq 0 \qquad A,B\leqslant(N-1)M/2.\end{gathered} \tag{13}$$

We cannot conclude that $A, B=(N-1)M/2$ because cancellations may occur in the eigenvalue equation (11). We show now, considering the limit $u\to\pm\mathrm{i}\infty$, that in the sector $Q=0$ we have $A=B=(N-1)M/2$. Only the case $u\to-\mathrm{i}\infty$, which fixes B, will be presented in detail, the case $u\to\mathrm{i}\infty$, which determines A, being completely analogous. After having observed that $[g(u)\bar{g}(u)]^M\sim(\mathrm{e}^{\mathrm{i}u})^{(N-1)M}$ when $u\to-\mathrm{i}\infty$, what we have to prove is that, for $u\to-\mathrm{i}\infty$, $\Lambda_{Q=0}(u)$ is finite and non-zero. Now, we have

$$W(n|-\mathrm{i}\infty)=\omega^{n(N-n)/2}$$
$$\bar{W}(n|-\mathrm{i}\infty)=\omega^{n(N+n)/2}$$

and

$$T_{n,n'}(-\mathrm{i}\infty)=\prod_{k=1}^{M}\omega^{n_k(n'_{k+1}-n'_k)}.$$

Notice that $T(-\mathrm{i}\infty)X=T(-\mathrm{i}\infty)$, hence eigenvalues $\Lambda_Q(-\mathrm{i}\infty)$ are 0 if $Q\neq 0$, but we cannot conclude yet that $\Lambda_{Q=0}(-\mathrm{i}\infty)\neq 0$. So we pass to the representation where the

X_k operators (and X) are diagonal

$$X_k|\sigma_k\rangle = \omega^{\sigma_k}|\sigma_k\rangle \qquad |n_k\rangle = \frac{1}{\sqrt{N}}\sum_{\sigma_k=0}^{N-1}\omega^{\sigma_k n_k}|\sigma_k\rangle.$$

In this basis we have

$$\begin{aligned} T(-\mathrm{i}\infty)_{\sigma,\sigma'} &= \sum_{n,n'}\langle\sigma|n\rangle\langle n, Tn'\rangle\langle n'|\sigma'\rangle \\ &= \frac{1}{N^M}\sum_{n,n'}\prod_{k=1}^{M}\omega^{n_k\sigma_k}\omega^{n_k(n'_{k+1}-n'_k)}\omega^{-\sigma'_k n'_k} \\ &= \sum_{n'}\prod_{k=1}^{M}\delta_{\sigma_k, n'_k-n'_{k+1}}\omega^{-\sigma'_k n'_k}. \end{aligned} \tag{14}$$

Here $\delta_{a,b}$ is understood mod N

$$\delta_{a,b} = \begin{cases} 1 & \text{if } a=b \bmod N \\ 0 & \text{otherwise.} \end{cases}$$

The sum can be performed after a change of summation variables $n'_1 = j_1$ $n'_2 = j_1+j_2, \ldots, n'_M = j_1+j_2+\ldots+j_M$ which gives

$$T_{\sigma,\sigma'}(-\mathrm{i}\infty) = N\delta_{\Sigma_{k=1}^M \sigma'_k,0}\delta_{\Sigma_{k=1}^M \sigma_k,0}\prod_{k=2}^{M}\omega^{\sigma'_k(\Sigma_{j=1}^{k-1}\sigma_j)}. \tag{15}$$

Since $\Sigma_{k=1}^M \sigma_k$ is the Z_N-charge of a state $|\sigma_1 \ldots \sigma_M\rangle$, (15) implies that $T_{\sigma,\sigma'}(-\mathrm{i}\infty)$ has non-zero elements in the $Q=0$ sector only. In such sector $T|_{Q=0}$ has an inverse

$$(T|_{Q=0})^{-1}_{\sigma',\tau} = \frac{1}{N^M}\delta_{\Sigma_{k=1}^M \sigma'_k,0}\delta_{\Sigma_{k=1}^M \tau_k,0}\prod_{k=2}^{M}\omega^{-\sigma'_k(\Sigma_{j=1}^{k-1}\tau_j)}.$$

Therefore all eigenvalues $\Lambda_{Q=0}(-\mathrm{i}\infty)$ are non-zero and $B=(N-1)M/2$. Inspecting the limit $u\to\mathrm{i}\infty$ we get $A=(N-1)M/2$. Now we can factorize (12) as

$$\Lambda_{Q=0}(u) = \left[\frac{1}{g(u)\bar{g}(u)}\right]^M (\mathrm{e}^{2\mathrm{i}u})^{-A}\rho(\mathrm{e}^{2\mathrm{i}u}-x_1^2)\ldots(\mathrm{e}^{2\mathrm{i}u}-x_{A+B}^2)$$

where ρ is a constant, all the zeros x_k^2 are non-vanishing and so can be written as $x_k^2 = \mathrm{e}^{2\mathrm{i}v_k}$. Finally

$$\begin{aligned} \Lambda_{Q=0}(u) &= \left[\frac{g(0)\bar{g}(0)}{g(u)\bar{g}(u)}\right]^M \prod_{k=1}^{L}\frac{\sin(u-v_k)}{\sin v_k} \\ L &= A+B = (N-1)M. \end{aligned} \tag{16}$$

The normalization has been fixed by $T(0) = 1_{\mathrm{id}}$.

Now we turn to the sectors $Q\neq 0$, and the symmetry under charge conjugation allows us to consider the sectors $Q=1,2,\ldots,(N-1)/2$ only. While we have not been able to obtain a proof like the one given above, one can show that, in the sector Q

(*a*) $A, B \leqslant (N-1)M/2 - Q \qquad Q=1,2,\ldots,(N-1)/2$

(*b*) $A, B \geqslant (N-1)M/2-(N-1)/2.$

The proof of (b) relies on a recursion relation for the transfer matrix derived from the set of relations in [10] and it will be discussed in the appendix. Here we show how to derive (a). Using the geometric series representation we write the BW

$$W(n)=W(n|-\mathrm{i}\infty)\prod_{j=1}^{n}\left[1+\sum_{q_j=1}^{+\infty}(\mathrm{e}^{-2\mathrm{i}u})^{q_j}\omega^{(1/2-j)q_j}(1-\omega^{2j-1})\right]$$

$$=W(n|-\mathrm{i}\infty)\left[1+\sum_{q=1}^{+\infty}W^{(q)}(n)\,\mathrm{e}^{-2\mathrm{i}uq}\right]$$

$$\bar{W}(n)=\bar{W}(n|-\mathrm{i}\infty)\prod_{j=1}^{n}\left[1+\sum_{q_j=1}^{+\infty}(\mathrm{e}^{-2\mathrm{i}u})^{q_j}\omega^{jq_j}(1-\omega^{1-2j})\right]$$

$$=\bar{W}(n|-\mathrm{i}\infty)\left[1+\sum_{q=1}^{+\infty}\bar{W}^{(q)}(n)\,\mathrm{e}^{-2\mathrm{i}uq}\right].$$

Inspection of the terms of order q shows that the coefficients $W^{(q)}(n)$, $\bar{W}^{(q)}(n)$, which must be polynomials in $\omega^{jn}, j=0,1,\ldots,N-1$, contain the terms $1,\omega^{n},\omega^{-n},\omega^{2n},\omega^{-2n},\ldots,\omega^{qn},\omega^{-qn}$ only. Next one considers the expansion of an arbitrary transfer matrix element in powers of $\mathrm{e}^{-2\mathrm{i}u}$. The coefficient of $\mathrm{e}^{-2\mathrm{i}uq}$ is a sum of terms like

$$T_{n,n'}(-\mathrm{i}\infty)\sum_{j_1\neq j_2\neq\ldots\neq j_n=1}^{M}\;\sum_{i_1\neq i_2\neq\ldots\neq i_m=1}^{M}\bar{W}^{(q_1)}(n_{j_1}-n'_{j_1})\ldots\bar{W}^{(q_n)}(n_{j_n}-n'_{j_n})$$

$$W^{(q'_1)}(n_{i_1}-n'_{i_1+1})\ldots W^{(q'_m)}(n_{i_m}-n'_{i_m+1})$$

where $q_1+\ldots+q_n+q'_1+\ldots+q'_m=q$. So the whole matrix element is a linear combination of terms, each of which looks like

$$T_{n,n'}(-\mathrm{i}\infty)\,\omega^{\pm p_1(n_{j_1}-n'_{j_1})\ldots\pm p_n(n_{j_n}-n'_{j_n})\pm p'_1(n_{i_1}-n'_{i_1+1})\ldots\pm p'_m(n_{i_m}-n'_{i_m+1})}$$

$p_1\leqslant q_1\ldots p_n\leqslant q_n, p'_1\leqslant q'_1\ldots p'_m\leqslant q'_m$. Substituting this into (14) shows that in the X representation all matrix elements in the sector $Q=q$ go to zero at least like $\mathrm{e}^{-2\mathrm{i}uq}$ and so must the eigenvalues because (8) and (11) hold no matter what representation has been chosen for $T(u)$. Hence $B\leqslant(N-1)M/2-Q$, $Q=1,2,\ldots,(N-1)/2$. Likewise, one proves $A\leqslant(N-1)M/2-Q$ by examining the $\mathrm{e}^{2\mathrm{i}u}$ expansion in the limit $u\to\mathrm{i}\infty$.

From (a) and (b) and (16) as well, we conclude that in the sectors $Q=0$ and $Q=(N-1)/2$

$$A=B=(N-1)M/2-Q$$

and the factorization in terms of sines can be carried out in (12) without the appearance of a phase $(\mathrm{e}^{2\mathrm{i}u})^{\pm(B-A)}$. We assume this to be true also for the others Q sectors, and arrive at the general form

$$\Lambda_Q(u)=\left[\frac{g(0)\bar{g}(0)}{g(u)\bar{g}(u)}\right]^{M}\prod_{k=1}^{L}\frac{\sin(u-v_k)}{\sin v_k}$$

$$L=(N-1)M-2Q\qquad Q=0,1,\ldots,(N-1)/2 \tag{17}$$

$$\Lambda_{N-Q}(u)=\Lambda_Q(u).$$

From this, the eitenvalue of H is easily found to be

$$E = \sum_{k=1}^{L} \cot v_k - 2M \sum_{j=1}^{(N-1)/2} \cot(\pi j/N). \tag{18}$$

4. Equations for the zeros

In this section, a system of Bethe-ansatz type equations for the zero $\{v_k\}$ will be found, using a recursion relation for the transfer matrix of the chiral Potts model which was derived in [9, 10]. We shall briefly describe the features of the self-dual chiral Potts model that are relevant here.

The transfer matrix, defined as in (7) but with the BW (3), (4), depends on two rapidity variables $p = (a_p, b_p)$ and $q = (a_q, b_q)$ subjected to the constraint (5). When p is kept fixed and q is varied, but still respecting (5), the commutation property (8) holds in the form

$$[T_{pq}, T_{pq'}] = 0.$$

Since p is thought fixed we will simply write T_q for T_{pq}. The constraint (5) is invariant under the two mappings

$$R(a, b) = (b, \omega a)$$

$$U(a, b) = (\omega a, b)$$

and the transfer matrix T_q satisfies [10]

$$\tilde{T}_q = \sum_{m=0}^{N-1} c_{m,q} T^{-1}_{U^m q} T_q T^{-1}_{U^{m+1} q} X^{-m-1} \tag{19}$$

where $\tilde{T} = TS$, $\bar{q} = (a_{\bar{q}}, b_{\bar{q}}) = UR^{-1}(a_q, b_q)$, and

$$c_{m,q} = \left[\left(\prod_{j=0}^{m-1} \frac{b_p - \omega^{j+1} a_q}{a_p - \omega^j a_q} \right) \left(\prod_{j=m+1}^{N-1} \frac{\omega(a_p - \omega^j a_q)}{b_p - \omega^{j+1} a_q} \right) \frac{N(b_q - b_p)(b_p - a_q)}{a_p b_p - \omega^m a_q b_q} \right]^M.$$

(The reader is cautioned that T used here is $\hat{T}$ of [10]. Since $[T_q, S] = 0$, (19) is equivalent to (4.40) of [10].) Equation (19) cannot be used as it stands: even though, in the non-chiral limit $\kappa \to 0$, T_q is reduced to $T(u)$ by applying the parametrization (6) and R corresponds to a shift $u \to u + \pi/2N$, U does not respect the parametrization (6) and it maps $T(u)$ into a different family $T'(u)$ constructed by choosing a different solution of $a^N + b^N = 0$. Nevertheless, even powers of U can be related to even powers of R by means of

$$T_{U^2 q} = A(p, r^2 q) X^{-1} T_{R^2 q}$$

$$A(p, q) = \left[\frac{(b_p - \omega a_q)(b_p - \omega^{-1} b_q)}{(\omega a_p - b_q)(a_p - a_q)} \right]^M.$$

For arbitrary even powers of U, the following relation is easily derived

$$T_{U^{2s} q} = \left[\prod_{j=1}^{s-1} A(p, U^{2j} R^{2(s-j)} q) \right] X^{-s} T_{R^{2s} q}.$$

When m is odd we write $U^m = U^{m+N}$ because $U^N = 1_{id}$ and if we restrict ourselves to odd N we are back to the previous case (this fact was already noticed in [10]). It is now lengthy but straightforward to express everything in terms of u. The result is

$$A(u)\tilde{T}\left(u+\frac{\pi}{2}\right)$$

$$= \sum_{j=0}^{(N-1)/2} p_{2j}(u) T^{-1}\left(u+\frac{j\pi}{N}\right) T^{-1}\left(u+\frac{j\pi}{N}+\frac{(N+1)\pi}{2N}\right) T(u)$$

$$+ \sum_{j=0}^{(N-3)/2} d_{2j+1}(u) T^{-1}\left(u+\frac{j\pi}{N}+\frac{(N+1)\pi}{2N}\right) T^{-1}\left(u+\frac{j\pi}{N}+\frac{\pi}{N}\right) T(u) \tag{20}$$

where

$$A(u) = \left[\prod_{j=0}^{(N-3)/2} \frac{\sin\left(u+\frac{(N+2j+1)\pi}{2N}\right)\sin\left(u+\frac{N-2j-2)\pi}{2N}\right)}{\sin\left(u+\frac{(N-2j-1)\pi}{2N}\right)\sin\left(u+\frac{(N+2j)\pi}{2N}\right)}\right]^M$$

$$p_{2s}(u) = \left[\frac{N \sin u \sin\left(u-\frac{\pi}{2N}\right)}{\sin\left(u+\frac{s\pi}{N}\right)\sin\left(u+\frac{s\pi}{N}-\frac{\pi}{2}\right)} \prod_{j=0}^{s-1} \frac{\sin^2\left(u+\frac{(2j+1)\pi}{2N}\right)}{\sin^2\left(u+\frac{j\pi}{N}\right)}\right.$$

$$\left. \times \prod_{j=s+(N+1)/2}^{N-1} \frac{\sin^2\left(u+\frac{j\pi}{N}\right)}{\sin^2\left(u+\frac{(2j+1)\pi}{2N}\right)}\right]^M$$

$$d_{2s+1}(u) = \left[\frac{N \sin u \sin\left(u-\frac{\pi}{2N}\right)}{\sin\left(u+\frac{(2s+1)\pi}{2N}\right)\sin\left(u+\frac{(2s+1)\pi}{2N}-\frac{\pi}{2}\right)} \prod_{j=0}^{s} \frac{\sin^2\left(u+\frac{(2j+1)\pi}{2N}\right)}{\sin^2\left(u+\frac{j\pi}{N}\right)}\right.$$

$$\left. \times \prod_{j=s+(N+1)/2}^{N-1} \frac{\sin^2\left(u+\frac{j\pi}{N}\right)}{\sin^2\left(u+\frac{(2j+1)\pi}{2N}\right)}\right]^M.$$

All inverses can be removed multiplying both sides with $\Pi_{j=1}^{N-1} T(u+2j\pi/2N)$. Since all matrices commute, equation (20) can be turned into a functional equation for each eigenvalue by acting with both sides on a common eigenvector. If v is any of the zeros $\{v_k\}$ of that eigenvalue, any choice

$$u = v - j\pi/N \qquad j = 1, 2, \ldots, N-1 \tag{21}$$

makes the LHS vanish and produces a set of equations for $\{v_k\}$. We simply observe here that the equations ensuing from different choices of j in (21) are all consistent and reduce to

$$\frac{\Lambda(v+(N+1)\pi/2N)}{\Lambda(v+(N-1)\pi/2N)}=-\left[\frac{\sin(v-\pi/2N)\sin(v+\pi/2)}{\sin v\sin(v-\pi/2N+\pi/2)}\right]^M. \tag{22}$$

A quicker way to get (22), as described in [9] for the general chiral Potts model, is to use (4.20) of [10], which reads for the self-dual case

$$\tau^{(2)}_{k,q}X^kT_{Uq}=\left[\frac{(b_p-\omega a_q)(a_pb_p-a_qb_q)}{b_p^2(a_p-a_q)}\right]^MT_q+\left[\frac{(b_p-b_q)(a_pb_p-\omega a_qb_q)}{b_q^2(a_p-b_q)}\right]^MT_{R^2q}.$$

Here $\tau^{(2)}_{k,q}$ is the transfer matrix of a related '2 by N' vertex model, and it commutes with T_q. Again, U must be turned into R and then, once the parametrization in u is made explicit, u is chosen to make the eigenvalue of T on the LHS vanish. Of course, the equation so derived coincides with (22). Replacing (17) in (22) we arrive at a set of Bethe-ansatz type equations

$$\prod_{k=1}^{L}\frac{\sin(v_j-v_k-(N-1)\pi/2N)}{\sin(v_j-v_k+(N-1)\pi/2N)}=(-1)^{M+1}\left[\frac{\sin(v_j-\pi/2N)}{\sin v_j}\right]^{2M} \tag{23}$$

$j=1,2,\ldots,L$. It is interesting to notice that the change of variables $v_k=\mathrm{i}\lambda_k+\pi/4N$ casts these equations into the form

$$\prod_{k=1}^{L}\frac{\sinh(\lambda_j-\lambda_k-\mathrm{i}\gamma)}{\sinh(\lambda_j-\lambda_k+\mathrm{i}\gamma)}=(-1)^{M+1}\left[\frac{\sinh(\lambda_j-\mathrm{i}s\gamma)}{\sinh(\lambda_j+\mathrm{i}s\gamma)}\right]^{2M}$$
$$\gamma=\frac{(N-1)\pi}{2N}\qquad s=\frac{1}{2(N-1)} \tag{24}$$

which is known to arise in the solution of the integrable XXZ spin chains [17, 18] and the (critical) generalized RSOS models [19]. While the general form is the same, some differences must be mentioned. In the XXZ chain s labels the representation of $SU(2)_q$ and takes the values $\frac{1}{2}, 1, \frac{3}{2},\ldots$ and γ, the anisotropy parameter, may change continuously. Moreover, in (24) the power of the RHS is *twice* the size of the chain and the number of unknowns can vary between $(N-1)M-(N-1)$ and $(N-1)M$ only and not between 0 and Ms. Similar equations were also found for the lattice version of the sine-Gordon model [20].

5. Free energy

Before we analyse (24), it is interesting to notice a property of the zeros $\{v_k\}$. Under a change of variables $u\to\pi/2N-u$ the transfer matrix transforms as

$$T^t(u)=T(\pi/2N-u)S$$

$\forall u\in C$ (T^t denotes matrix transposition). Moreover, if $u\in R$ the BW are real, so $T^t(u)=T^\dagger(u)$. All this implies that $[T^\dagger(u), T(u)]=0$ and

$$\Lambda^*(u)=\Lambda(\pi/2N-u)\,\mathrm{e}^{-\mathrm{i}P} \tag{25}$$

when u is real (* denotes complex conjugation). Inserting (17) into (25) yields

$$\prod_{k=1}^{L}\frac{\mathrm{e}^{-2\mathrm{i}u}-x_k^{*2}}{x_k^{*2}-1}=\mathrm{e}^{-\mathrm{i}P}\omega^{L/4}\prod_{k=1}^{L}\frac{\mathrm{e}^{-2\mathrm{i}u}-\omega^{-1/2}x_k^2}{x_k^2-1}. \tag{26}$$

Although (26) has been derived for u real, the two sides are meromorphic functions in $z=e^{-2iu}$ and the equality must hold in the whole complex plane, so that the set of zeroes of any eigenvalue has the conjugation property

$$\{x_k^{*2}\}=\{\omega^{-1/2}x_k^2\}.$$

Two possibilities can occur: either $x_k^{*2}=\omega^{-1/2}x_k^2$ or two zeros $x_k, x_{k'}$ are paired $x_k^{*2}=\omega^{-1/2}x_{k'}^2$. Writing $\lambda=\lambda^R+i\lambda^I$, we have in the former case

$$\lambda_k^I=0 \qquad \text{or} \qquad \lambda_k^I=\pi/2 \tag{27}$$

while the latter yields

$$\lambda_k^R=\lambda_{k'}^R \qquad \lambda_k^I=-\lambda_{k'}^I. \tag{28}$$

Notice that the periodicity of (17) and (24) allows one to take $-\pi/2<\lambda_k^I\leqslant\pi/2$.

Several problems arise in the analysis of (23) or, equivalently, (24). A preliminary study for very small chains shows that (23) produces spurious solutions which do not correspond to any eigenvalue of H (or $T(u)$). Some of these solutions give a complex energy (18) and must obviously be discarded. Other solutions have two identical roots, $v_k=v_j$, and do not appear in the spectrum when H is diagonalized on small chains. Finally, if $\{v_k\}$ is a solution of (23), any permutation of the roots would give a distinct solution with the same energy, but, again, this degeneracy does not appear in the spectrum. The last two remarks suggest that the wavefunction might acquire a phase under a permutation of the roots $\{v_k\}$ (this is known to be the case for models solved with the Bethe-ansatz [21]). The same phenomenon was observed in [15].

On the other hand, (24) is usually studied in the framework of the string hypothesis [22], according to which, in the thermodynamic limit, the solutions group into complexes

$$\lambda_{\alpha,k}^{(n,v)}=\lambda_\alpha^{(n,v)}+\frac{\gamma}{2}(n+1-2k)i+\frac{(1-v)\pi}{4}i+\delta_{\alpha,k}^{(n,v)} \qquad k=1,2,\ldots,n. \tag{29}$$

$\lambda_\alpha^{(n,v)}$ is the centre of the string, n its length, $v=\pm1$ its parity and δ, a deviation from the perfect string behaviour, is supposed to vanish when $M\to\infty$. It is convenient to label strings by an integer j, shorthand notation for (n_j, v_j). The standard method [22] to obtain equations for the centres of the strings is to multiply (24) over different roots in the same string and then take the logarithm of both sides, reducing it to

$$\frac{1}{2\pi}t_j(\lambda_\alpha^j)-\frac{1}{2\pi M}\sum_k\sum_{\beta=1}^{M_k}\Theta_{jk}(\lambda_\alpha^{(j)}-\lambda_\beta^{(k)})=\frac{I_\alpha^{(j)}}{M} \tag{30}$$

where M_k is the number of k-strings. In writing (30), the following definitions have been used

$$t_j(\lambda)=2\sum_{l=1}^{n_j}\phi(\lambda, n_j+2s-2l+1, v_j)$$

$$\Theta_{jk}(\lambda)=\phi(\lambda, n_j+n_k, v_jv_k)+\phi(\lambda, |n_j-n_k|, v_jv_k)+\sum_{l=1}^{\min(n_j,n_k)-1}2\phi(\lambda, |n_j-n_k|+2l, v_jv_k)$$

and

$$\phi(\lambda, n, v)=\begin{cases}2v\tan^{-1}(\cot(n\gamma/2)^v\tanh(\lambda)) \\ 0 \qquad \text{if } n\gamma=q\pi \qquad q\in\mathbb{Z}.\end{cases}$$

The $I_\alpha^{(j)}$ are integers or half-odd.

A study of (30) requires a preliminary analysis of which lengths and parities can appear. In [23] the problem has been investigated with regard to the integrable XXZ chains. When the number of roots, L in our notation, is kept fixed as $M \to \infty$, a rather simple set of inequalities on the allowed lengths and parities can be derived. Unfortunately, as (17) shows, L goes to infinity with M and this makes the problem much more difficult. Besides that, a study of small chains shows that complex pairs (λ, λ^*) may appear which are not in the form (29). This is known to happen for the XXZ chains [24], even in the isotropic case [25]. In fact, the restrictions (27), (28) do not rule out this possibility. Finally, and, again, unlike the XXZ chains, the contribution to (18) from a single string is not necessarily real. One may then wonder whether some solutions of (30) have to be discarded or the string centres that solve it automatically arrange themselves so that the energy is real.

All these questions are still open and currently under investigation. Here we will limit ourselves to show that, by filling the real axis with $(N-1)$-strings of alternating parity $v=(-1)^{(N+1)/2}$, one recovers the free energy originally obtained in [1] with the matrix inversion method, and later rederived in another way [28]. It is convenient to rewrite (30) in a slightly different form [26]. Define a Z function

$$Z_j(\lambda)=\frac{1}{2\pi}\,t_j(\lambda)-\frac{1}{2\pi M}\sum_k\sum_{\beta=1}^{M_k}\Theta_{jk}(\lambda-\lambda_\beta^{(k)}). \tag{31}$$

Then (30) becomes

$$Z_j(\lambda_\alpha^{(j)})=\frac{I_\alpha^{(j)}}{M}. \tag{32}$$

In the thermodynamic limit, the centres of j-strings fill the real axis (or a region of the real axis) with density $\rho_j(\lambda)$

$$\rho_j(\lambda_\alpha^{(j)})=\lim_{M\to\infty}\frac{1}{M(\lambda_{\alpha+1}^{(j)}-\lambda_\alpha^{(j)})}.$$

If the $I_\alpha^{(j)}$ are arranged in a sequence without jumps

$$Z_j'(\lambda)=\pm\rho_j(\lambda) \tag{33}$$

where the sign of the RHS is + (−) if increasing $I_\alpha^{(j)}$ correspond to increasing (decreasing) $\lambda_\alpha^{(j)}$. Obviously, (33) only holds in the region of the real axis that is actually filled with j-roots.

Perron–Frobenius theorem guarantees that, for all finite M, the ground state of H belongs to the sector $Q=0$, $P=0$. Consider a state with M strings of length $N-1$ and parity

$$v=(-1)^{(N+1)/2} \tag{34}$$

hence $M(N-1)$ roots in all. It is easily derived from (31), (32) that $I_\alpha^{(N-1)}/M$ can vary between $Z_{N-1}(+\infty)=-\frac{1}{2}$ and $Z_{N-1}(-\infty)=\frac{1}{2}$. The $I_\alpha^{(N-1)}$ are half-odd when M is even and integers when M is odd, and we have M of them, so we conclude that they form a closely packed sequence and $(N-1)$-strings fill the real axis with a density $\rho_{N-1}(\lambda)$, solution of the integral equation

$$-\rho_{N-1}(\lambda)=\frac{1}{2\pi}\,t_{N-1}'(\lambda)-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}\mu\,\Theta_{N-1,N-1}'(\lambda-\mu)\rho_{N-1}(\mu) \tag{35}$$

where v is understood to be chosen as in (34). This equation can be solved taking the Fourier transform. Our conventions are

$$f(\lambda)=\frac{1}{2\pi}\int \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}k\lambda}\tilde{f}(k) \qquad \tilde{f}(k)=\int \mathrm{d}\lambda\, \mathrm{e}^{\mathrm{i}k\lambda}f(\lambda).$$

Moreover one needs

$$\phi'(\lambda,n,v)=\frac{\mathrm{d}}{\mathrm{d}\lambda}\phi(\lambda,n,v)=\frac{2v\sin(n\gamma)}{\cosh 2\lambda - v\cos(n\gamma)}$$

$$\tilde{\phi}'(k,n,v)=\int \mathrm{d}\lambda\, \mathrm{e}^{\mathrm{i}k\lambda}\phi'(\lambda,n,v)=2\pi\frac{\sinh(k\pi/2-kx\pi)}{\sinh(k\pi/2)}$$

where $x=\{n(N-1)/4N+(1-v)/4\}$ and $\{\}$ denotes the fractional part. The solution of (35) is then straightforward if one uses the sum rules for hyperbolic functions [27]

$$\tilde{\rho}_{N-1}(k)=\frac{1}{\cosh(k\pi/4N)} \qquad \rho_{N-1}(\lambda)=\frac{2N}{\pi\cosh(2N\lambda)}. \tag{36}$$

The free energy per site of the 2D model is

$$f(u)=\lim_{M\to\infty}-\frac{1}{M}\ln\Lambda(u)$$

if $\Lambda(u)$ is the largest eigenvalue of the transfer matrix. Under the assumption that the state (36) is indeed the ground state of H, we find in the physical region $0\leqslant u\leqslant \pi/2N$

$$f(u)=-\ln\left(\frac{g(0)\bar{g}(0)}{g(u)\bar{g}(u)}\right)-\frac{1}{2}\int_{-\infty}^{+\infty}\mathrm{d}\lambda\,\rho_{N-1}(\lambda) \times\sum_{j=1}^{N-1}\ln\frac{\cosh 2\lambda-\cos\left(2u+\frac{N-1}{2N}\pi(N-2j)+\frac{\pi(1-v)}{2}-\frac{\pi}{2N}\right)}{\cosh 2\lambda-\cos\left(\frac{N-1}{2N}\pi(N-2j)+\frac{\pi(1-v)}{2}-\frac{\pi}{2N}\right)} \tag{37}$$

where the sum comes from taking the product of roots in the same string and $\rho_{N-1}(-\lambda)=\rho_{N-1}(\lambda)$ has been used. One writes then

$$\rho_{N-1}(\lambda)=\frac{1}{2\pi}\frac{\partial}{\partial\lambda}\int\frac{\mathrm{i}}{k}\mathrm{e}^{-\mathrm{i}k\lambda}\tilde{\rho}_{N-1}(k)$$

and integrates (37) by parts. After the integral over λ is evaluated by contour integration, one arrives at

$$f(u)=-\ln\left(\frac{g(0)\bar{g}(0)}{g(u)\bar{g}(u)}\right)+f_0(u)=-\ln G(u)+f_0(u)$$

$$f_0(u)=-2\int_{-\infty}^{+\infty}\frac{\mathrm{d}k}{k}\frac{\left(\begin{array}{c}\sinh(ku)\sinh(k(\pi/2N-u))\\ \times\sinh(k\pi(N-1)/2N)\cosh(k\pi(N+1)/2N)\end{array}\right)}{\cosh(k\pi/2N)\sinh(k\pi)\sinh(k\pi/N)}.$$

After a change of variables $k=Nx, f_0(u)$ is broken into two parts

$$f_0(u)=-\frac{1}{2}\int_{-\infty}^{+\infty}\frac{\mathrm{d}x}{x}\frac{\sinh(Nxu)\sinh(Nx(\pi/2N-u))\sinh(x\pi(N-1)/2)}{\cosh(x\pi/2)\sinh(x\pi/2)\sinh(x\pi N/2)} -\frac{1}{2}\int_{-\infty}^{+\infty}\frac{\mathrm{d}x}{x}\frac{\sinh(Nxu)\sinh(Nx(\pi/2N-u))\sinh(x\pi(N-1)/2)}{\cosh^2(x\pi/2)\cosh(Nx\pi/2)}. \tag{38}$$

The first term on the RHS of (38) can be explicitly calculated and it exactly cancels $-\ln G(u)$, so that $f(u)$ is given by the second term on the RHS of (38), which is precisely the result of [1].

As a final remark, we observe that there exists a deep relation between the $N=3$ case (Potts model) and the critical RSOS model with parameters $p=1, q=1, r=6$ studied in [19]. In fact, the recursion relation (20) for the renormalized transfer matrix

$$T^N(u)=T(u)\left[\frac{g(u)\bar{g}(u)}{g(0)\bar{g}(0)}\right]^M \tag{39}$$

is written as

$$\begin{aligned} T^N(u)\tilde{T}^N(u+\pi/6)T^N(u+\pi/3) &=\left[\frac{\sin u\sin(u+\pi/3)}{\sin^2(\pi/6)}\right]^{2M}T^N(u)\\ &\quad+\left[\frac{\sin(u-\pi/6)\sin(u+\pi/6)}{\sin^2(\pi/6)}\right]^{2M}T^N(u+\pi/3)\\ &\quad+(-1)^M\left[\frac{\sin u\sin(u+\pi/6)}{\sin^2(\pi/6)}\right]^{2M}T^N(u+2\pi/3). \end{aligned} \tag{40}$$

Take M even and consider the sector $P=0$, where $\tilde{T}=T$. Then, from (3.19′) and (3.19″) of [19] an identical recursion relation can be derived for the transfer matrix T_{RSOS} of the model with parameters $p=1, q=1, r=6$, at least in the sector where the operator Y defined therein has eigenvalue 1, and provided that the size of the chain is doubled.

This is not sufficient to conclude that there is a one-to-one correspondence of the eigenvalues, but we may conjecture that the largest eigenvalues coincide. This can be checked for the leading order in the thermodynamic limit. Observe first that the free energy per site was computed in [19] after renormalizing the transfer matrix (here we take into account the doubling of the chain length)

$$T_{\mathrm{RSOS}}(u)\to T^N_{\mathrm{RSOS}}=\left[\frac{\sin(u+\pi/6)}{\sin(\pi/6)}\right]^{2M}T_{\mathrm{RSOS}}(u) \tag{41}$$

and it was found to be (equation (5.8) of [19])

$$f_{\mathrm{RSOS}}(u)=\int_{-\infty}^{+\infty}\frac{\mathrm{d}x}{x}\,\frac{\sinh(12ux/\pi)\sinh x\sinh 3x}{\sinh 6x\sinh 2x}. \tag{42}$$

Denoting by $f(u)$ the free energy per site of the Potts model, from (39) and (41) we expect the following equality to hold

$$f(u)=2f_{\mathrm{RSOS}}(u)-\ln\left(\frac{\sin(u+\pi/6)\sin(\pi/3)}{\sin(\pi/3-u)\sin(\pi/6)}\right). \tag{43}$$

Using (38) and (42) it is easily checked that (43) is correct. The relation between free energies of the two models was already pointed out in [19].

Appendix

We derive here a recursion relation quadratic in the transfer matrix that allows us to put a lower bound on A and B. In the homogeneous case $p=p'$, i.e. when all vertical rapidities are equal, the following relation holds [10]

$$\lambda_q^{(k,l)}\tilde{T}_q T_{\bar{q}(k,l)} = \bar{H}_{pq}^{(j)}\tau_{k,q}^{(j)} + H_{pq}^{(j)}\tau_{k-j,\bar{q}(k,l)}^{(N-j)}. \tag{44}$$

Here k, l, j are integers $(j = k+l)$, $\tau_{k,q}^{(j)}$ is a family of transfer matrices for a vertex model with N-valued (j-valued) spins on the vertical (horizontal) bonds. Moreover, $\bar{q}(k,l) = U^{k-l+1}R^{2l-1}q = U^{k-l+1}R^{2l-1}(a_q, b_q)$ and λ, H, $\bar{H}$ are functions of a_q, b_q, a_p, b_p. For further details the reader is referred to the original paper.

In [10] it was noticed that

$$\tau_{k,q}^{(1)} = X^{-1}$$

and that

$$\tau_{k,q}^{(j)} = \tau_{k,\bar{q}}^{(j)}$$

where $\bar{q} = \bar{q}(0,0) = UR^{-1}q$ is the 'conjugate' of q. We look for a rapidity q' such that

$$\bar{q}'(k,l) = U^{k-l+1}r^{2l-1}q' = \text{conjugate of } \bar{q}(k,l) = UR^{-1}\bar{q}(k,l).$$

Such rapidity is easily found to be $q' = U^{2k-2l+1}R^{2l-2k-1}q$. Therefore, after taking $j=1$, the τ matrices can be eliminated subtracting from (44) the same equation with $q \to q'$ and for our purposes it is sufficient to consider the case $k=0$, $l=1$. In the resulting equation, the mapping U is eliminated as explained in section 4 (again, only for N odd is this possible) and everything is parametrized with (6). The final result is

$$[-\cos^2 u]^M \tilde{T}(u)T(u+\pi/2N) - [\sin^2 u]^M \tilde{T}(u+\pi/2)T(u+\pi/2+\pi/2N)$$
$$= \left[-\frac{N\cos(Nu)}{\sin(Nu)}\cos u \sin u\right]^M - \left[\frac{N\sin(Nu)}{\cos(Nu)}\cos u \sin u\right]^M. \tag{45}$$

Introduce the variable $z = e^{iu}$. When $u \to -i\infty$, $z \to \infty$ and the RHS of (45) grows like

$$\sim z^{2M-2N}.$$

On the other hand, the LHS cannot grow faster than

$$\sim z^{2M}\Lambda^2(z) \sim z^{2M}(z^{2B-M(N-1)})^2$$

so we must have

$$4B - 2M(N-1) \geqslant -2N.$$

Since B is integer and N is odd, the inequality can be improved to

$$B \geqslant M(N-1)/2 - (N-1)/2.$$

If $u \to i\infty$, $z \to 0$ and we get in the same manner

$$A \geqslant M(N-1)/2 - (N-1)/2.$$

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